PMM U.S.S.R., Vol.47,No.2,pp.188-196,1984
Printed in Great Britain

0021-8928/ 84 $10.00+0.00

UDC 539.3

STATICALLY ADMISSIBLE STRESS FIELDS IN INCOMPRESSIBLE MEDIA*

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A set of statically admissible stress fields must often be considered in problems of the mechanics of a continuous medium. In particular, the extremums should be sought in this set in conformity with the Castigliano principle for the static limit coefficient of the theory of ideal plasticity. Two different sets of statically admissible stresses are used for incompressible media. Their interrelation is known only for kinematic conditions on the whole boundary of the body. An analogous relation is established in this paper for the case of mixed boundary conditions, and the possibility of its utilization is discussed.

Let a continuous medium fill the domain Ω in R^n $(n=2,3)$, and let it be subjected to mass forces with density $\mathbf{f}$ given in Ω, and a surface load with density $\mathbf{q}$ given on a part S_q of the boundary of the domain Ω. Furthermore, let

$$S_v = \partial\Omega \setminus \bar{S}_q,\ S_q = \partial\Omega \setminus \bar{S}_v \tag{0.1}$$

where the bar denotes the closure in R^n, and the velocity is given on S_v (for instance, S_v is clamped).

A stress field equilibrating the load $(\mathbf{f}, \mathbf{q})$ is called statically admissible; the equilibrium conditions in the domain Ω and on its boundary can be written in the form of equations for the principle of virtual velocities /1/

$$\int_\Omega \sigma\cdot\mathbf{e}\,dx - \int_\Omega \mathbf{f}\mathbf{v}\,dx - \int_{S_q} \mathbf{q}\mathbf{v}\,ds = 0 \quad \forall \mathbf{v} \in \mathbf{V} \tag{0.2}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right), \quad i, j = 1, 2, \ldots, n$$

where $\mathbf{V}$ is the set of virtual (trial) velocity fields, and x^i are Cartesian coordinates in R^n.

For incompressible media there are two natural possibilities for the selection of $\mathbf{V}: \mathbf{V}^1 = \mathbf{V}^1(\Omega, S_v)$ is the set of smooth, i.e., belonging to $\mathbf{C}^\infty(\Omega)$, solenoidal velocity fields that vanish near (or on) S_v and $\mathbf{V}^2 = \mathbf{V}^2(\Omega, S_v)$, the set that is defined analogously but without the solenoidality requirement. In this connection, the following question occurs. Let a stress field τ satisfy the equilibrium conditions (0.2) with $\mathbf{V} = \mathbf{V}^1$ (then any stress field $\tau + p\mathbf{g}$ also satisfies it, where p is an arbitrary pressure field and $\mathbf{g}$ is a metric tensor); could a pressure field p be found such that $\tau + p\mathbf{g}$ would satisfy the complete equilibrium conditions, i.e., (0.2) with $\mathbf{V} = \mathbf{V}^2$?

This question was apparently first studied in /2/ in application to hydromechanics problems in which the kinematic boundary conditions $(\partial\Omega = S_v)$ are given on the whole boundary $\partial\Omega$ It turns out that such a pressure field can be found. Another case, of mixed boundary conditions, is also of interest. It is shown below that a suitable pressure field can be found even in this case. The role of this assertion is discussed in Sect.5.

1. Formulation of the problem. Plan of the solution. Let $\mathbf{s}$ $(s_{ij} = s_{ji};\ s_{ij} \in L_2(\Omega);\ i, j = 1, 2, \ldots, n)$ be some stress field satisfying (0.2) with $\mathbf{V} = \mathbf{V}^2$ and let Σ be a set of stress fields equilibrating the load $\mathbf{f} = 0, \mathbf{q} = 0$ (such stress fields are called self-equilibrating). Then the set of stress fields, statically admissible for the load $(\mathbf{f}, \mathbf{q})$ can be represented in the form $\Sigma + \mathbf{s}$. If $\mathbf{s}_1$ and $\mathbf{s}_2$ are two stress fields equilibrating the load $(\mathbf{f}, \mathbf{q})$, then their difference is evidently self-equilibrated, and therefore, $\Sigma + \mathbf{s}_1 = \Sigma + \mathbf{s}_2$. The study of the set of statically admissible stress fields therefore actually reduces to a study of Σ.

In conformity with the two possibilities for the selection of $\mathbf{V} - \mathbf{V}^1(\Omega, S_v)$ and $\mathbf{V}^2(\Omega, S_v)$, there are also two sets of self-equilibrated stress fields

*Prikl.Matem.Mekhan.,Vol.47,No.2,pp.228-237,1983

$$\Sigma^{\varkappa}=\Sigma^{\varkappa}(\Omega,S_v)=(\mathbf{E}(\mathbf{V}^{\varkappa}))^{\circ}=$$
$$\Big\{\sigma:\sigma_{ij}=\sigma_{ji},\ \sigma_{ij}\in L_2(\Omega)\ \ (i,j=1,2,\ldots,n);\ \int\limits_{\Omega}\sigma\cdot\mathbf{e}\,dx=0,$$
$$\forall\mathbf{e}\in\mathbf{E}(\mathbf{V}^{\varkappa})\Big\}\quad(\varkappa=1,2)$$

where $\mathbf{E}(\mathbf{V}^{\varkappa})$ is the set of strain rates corresponding to the velocity fields from $\mathbf{V}^{\varkappa}=\mathbf{V}^{\varkappa}(\Omega, S_v)$.

It is required to establish that for every τ a pressure field p is found from Σ^1 in $L_2(\Omega)$ such that $\tau+pg$ is in Σ^2.

Let us examine a plan for solving this problem by first assuming that the boundary of the domain and the initial stress field τ are smooth.

First step. Since τ belongs to Σ^1, then the equality, in particular

$$\int\limits_{\Omega}\tau_{ij}\frac{\partial v_i}{\partial x^j}\,dx=0$$

is satisfied for all smooth solenoidal fields $\mathbf{v}$ with compact carriers in Ω ($\mathbf{v}\in\mathbf{D}(\Omega)$, $\operatorname{div}\mathbf{v}=0$).

By virtue of the results in /3/, there follows from this that

$$\partial\tau_{ij}/\partial x^j=\partial t/dx^i \tag{1.1}$$

where t is a smooth function because of the smoothness of τ. If $\partial\Omega=S_v$, then the construction terminates here; the desired pressure field is $p=-t$.

Second step. Since $\tau^{\circ}=\tau-tg$ lies on Σ^1 together with τ, then for any $\mathbf{v}$ from $\mathbf{V}^1$ ($\mathbf{v}\in C^{\infty}(\bar\Omega)$, $\mathbf{v}|_{S_v}=0$, $\operatorname{div}\mathbf{v}=0$)

$$\int\limits_{\Omega}\tau_{ij}^{\circ}\frac{\partial v_i}{\partial x^j}\,dx=0 \tag{1.2}$$

Because of (1.1) we have $\partial\tau_{ij}^{\circ}/\partial x^j=0$, then we find from (1.2) by the Stokes formula that

$$\int\limits_{\partial\Omega}\gamma\mathbf{v}\,ds=0\quad\forall\mathbf{v}\in\mathbf{V}^1;\ \gamma_i=\tau_{ij}^{\circ}\nu_j \tag{1.3}$$

(ν is the unit external normal to $\partial\Omega$).

Third step. Now let $\mathbf{u}$ be any field from $\mathbf{V}^2$ ($\mathbf{u}\in C^{\infty}(\bar\Omega)$, $\mathbf{u}|_{s_v}=0$), and $\mathbf{u}_0$ some smooth field for which

$$\mathbf{u}_0|_{S_v}=0,\quad\int\limits_{\partial\Omega}\mathbf{u}_0\nu\,ds=1$$

(such a $\mathbf{u}$ is found if $S_q\neq\varnothing$).

Then for

$$\mathbf{v}=\mathbf{u}-\mathbf{u}_0\int\limits_{\partial\Omega}\mathbf{u}\nu\,ds \tag{1.4}$$

the following relationships are satisfied

$$\mathbf{v}|_{S_v}=0,\quad\int\limits_{\partial\Omega}\mathbf{v}\nu\,ds=0 \tag{1.5}$$

Let us now consider the trace $\mathbf{v}|_{\partial\Omega}$. Because of the second of the conditions (1.5) it has a smooth solenoidal continuation on $\Omega-\mathbf{v}_s$ from $\mathbf{V}^1$. Then it follows from (1.3)

$$\int\limits_{\partial\Omega}\gamma\mathbf{v}_s\,ds=0$$

which by utilizing (1.4) results in the relationship

$$\int\limits_{\partial\Omega}\gamma\mathbf{u}\,ds=c_0\int\limits_{\partial\Omega}\nu\mathbf{u}\,ds\quad\forall\mathbf{u}\in\mathbf{V}^2;\quad c_0=\int\limits_{\partial\Omega}\gamma\mathbf{u}_0\,ds \tag{1.6}$$

There remains to set $\sigma = \tau - (t + c_0)\,g$ and to use (1.6) and the Stokes formula to see that for any u from V^2

$$\int_{\partial\Omega} \sigma_{ij} \frac{\partial u_i}{\partial x^j}\, dx = 0$$

Therefore, σ lies in Σ^2 and the required pressure is $p = -(t + c_0)$.

This plan will later be realized under weakened assumptions on the smoothness of $\partial\Omega$ and τ. Necessary for this is a certain preparatory operation since the construction produced is fraught with a number of difficulties in the unsmooth case.

Namely, if the field τ is not smooth, then even the t occurring in the first step can be considered just as a generalized function. This difficulty was overcome in /2/. The results obtained in /2/ for hydromechanics problems (in which $\partial\Omega = S_v$) carry over automatically to the general case for $\Omega = S_v$ (Theorem 5.1).

Furthermore, utilization of the Stokes formula is needed in the foundation for unsmooth τ. This foundation is given in Sect.4.

Finally, if the boundary $\partial\Omega$ is not assumed smooth, then the continuation of v_s considered in the third step is not generally smooth. In this case it cannot belong to V^1, which does not, in turn, permit direct utilization of the relationship (1.3) for it in order to obtain (1.6). Therefore, V^1 must be expanded to a certain set $\bar{V}^1$ such that firstly the set of self-equilibrated stress fields would remain as before $\Sigma^1 = (E(V^1))^c = (E(\bar{V}^1))^c$, secondly the Stokes formula could be applied in (1.2) to derive (1.3) for any v from $\bar{V}^1$, and thirdly, the solenoidal continuation of v_s in the third step would belong to $\bar{V}^1$. Such an expansion is considered in Sect.3. The expansion of the set V^2 is examined first in Sect.2.

2. Trial velocity fields.

Since $\sigma_{ij} \in L_2(\Omega)$ for the stress fields under consideration, then compliance with the equilibrium conditions

$$\int_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x^j}\, dx = 0$$

for all v from V^1 (from V^2) is equivalent to satisfying them for all v from $\bar{V}^1$ (from $\bar{V}^2$), where $\bar{V}^1(\bar{V}^2)$ is the closure of $V^1(V^2)$ in $H^1(\Omega)$. Here $H^1(\Omega)$ and the $H^{1/2}(\partial\Omega)$ utilized later are Sobolev spaces whose properties have been studied well /4–7/. The closures $\bar{V}^1$, $\bar{V}^2$ turn out to be suitable expansions of V^1, V^2.

We also note that the set V^2 of the trial velocity fields can itself be defined by two methods, as all possible smooth velocity fields v in $\bar{\Omega}$ that vanish on or near $\bar{S}_v$ or close to $\bar{S}_v$ (the latter means that the distance from the carrier $\operatorname{supp} v$ to $\bar{S}_v$ is positive). Later, not to distinguish these cases, we consider their corresponding sets of self-equilibrated stress fields to coincide. Coincidence is assured if $U^2 = W^2$, where

$$\begin{aligned} U^2 &= U^2(\Omega, S_v) = [\{u \in C^\infty(\bar{\Omega}) : \rho(\operatorname{supp} u, \bar{S}_v) > 0\}]_{H^1(\Omega)} \\ W^2 &= W^2(\Omega, S_v) = \{w \in H^1(\Omega) : w|_{S_v} = 0\} \end{aligned} \tag{2.1}$$

($\rho(A, B)$ is the distance between the sets A and B in R^n). Lemma 2.1 will yield the sufficient conditions for the equality $U^2 = W^2$ which will henceforth be used.

A certain regularity of $\partial\Omega$ and S_v is required for the proof of Lemma 2.1. We will consider that Ω is a bounded domain of class C^1. This means that $\bar{\Omega}$ can be covered by a finite number of domains U_i on which mappings φ_i are defined that are continuously differentiable and have continuously differentiable inverses. A standard cylinder in R^n is the pattern for the domain U_i with the mapping φ_i, and a sphere in R^{n-1} is the pattern of $U_i \cap \partial\Omega$ (if the intersection is non-empty). Let us note that continuously differentiable functions α_i exist in R^n with carriers in U_i, that accomplish the partition of unity in $\bar{\Omega}: \sum \alpha_i|_{\bar{\Omega}} = 1$.

We shall call the part S_v of the boundary $\partial\Omega$ regular if the map (U_i, φ_i) can be selected such that the set G_i the complement to the closure of the set $\varphi_i(U_i \cap S_v)$ in R^n, is the domain for each point of the boundary ∂G_i of which a neighborhood U in R^{n-1} and a direction ξ exist such that for any sufficiently small shift in the direction ξ the set $\bar{G}_i \cap U$ will not emerge beyond the limits of the domain G_i. This latter property is satisfied, for instance, for a strictly Lipschitzian domain G_i.

Lemma 2.1. Let Ω be a bounded domain of class C^1, and S_v the regular part of its boundary. Then $U^2(\Omega, S_v) = W^2(\Omega, S_v)$.

The embedding of U^2 in W^2 is evident. Moreover, every w from W^2 belongs to U^2. For the proof, the traces $w|_{\partial\Omega}$ in $H^{1/2}(\partial\Omega)$ can be approximated by functions v_ε that vanish on a circle in $\partial\Omega$ of the set S_v. By virtue of the regularity of S_v for any $\varepsilon > 0$ the estimate $\|\mathbf{f}_\varepsilon\| < \varepsilon$ can be assured for the function $\mathbf{f}_\varepsilon = \mathbf{w}|_{\partial\Omega} - \mathbf{v}_\varepsilon$ in $\mathbf{H}^{1/2}(\partial\Omega)$. In the domain Ω of the class C^1 any function $\mathbf{g}$ from $\mathbf{H}^{1/2}(\partial\Omega)$ has the continuation $\mathbf{g}^c$

$$\begin{gathered} \mathbf{g}^c \in \mathbf{H}^1(\Omega), \quad \mathbf{g}^c|_{\partial\Omega} = \mathbf{g} \\ \|\mathbf{g}^c\|_{\mathbf{H}^1(\Omega)} \leqslant c \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\partial\Omega)} \end{gathered} \tag{2.2}$$

The function $\mathbf{w}_\varepsilon = \mathbf{w} + (\mathbf{f}_\varepsilon)^c$ approximates $\mathbf{w}$

$$\|\mathbf{w} - \mathbf{w}_\varepsilon\|_{\mathbf{H}^1(\Omega)} \leqslant c\varepsilon$$

(c is independent of $\mathbf{w}$) and vanishes in a certain neighborhood (in $\partial\Omega$) of the set S_v. It remains to apply the following assertion to $\mathbf{w}_\varepsilon$.

Lemma 2.2. Let Ω be a bounded strictly Lipschitzian domain, S a closed subset in $\partial\Omega$, and $\mathbf{u}$ a function from $\mathbf{H}^1(\Omega)$, whose trace vanishes in a neighborhood S' in $\partial\Omega$ of the set S. Then for any $\varepsilon > 0$ there exists a smooth function $\mathbf{u}_\varepsilon$ from $\mathbf{C}^\infty(\overline{\Omega})$ which vanishes in a certain neighborhood of the set S in $\overline{\Omega}$ and which approximates the function $\mathbf{u}$:

$$\|\mathbf{u} - \mathbf{u}_\varepsilon\|_{\mathbf{H}^1(\Omega)} < \varepsilon$$

By using the partition of unity, the proof of Lemma 2.2 is reduced to confirming it for a star domain relative to a sphere with center at zero. Furthermore, it is sufficient to consider a suitable continuation u^c of the function u and a sequence of averages of the functions u_λ ($u_\lambda(x) = u^c(\lambda x)$, $\lambda > 1$) as $\lambda \to 1$.

Finally, we note that if $\mathbf{U}^2 = \mathbf{W}^2$, then evidently $\mathbf{U}^2 = \overline{\mathbf{V}}^2 = \mathbf{W}^2$ independently of whether the trial velocity fields should vanish near or on $\overline{S}_v$ is required in the definition of $\mathbf{V}^2$.

3. Solenoidal velocity fields. For solenoidal fields in Ω we set up the analog to Lemma 2.1., i.e., agreement between the sets $\mathbf{U}^1$ and $\mathbf{W}^1$, where

$$\begin{gathered} \mathbf{U}^1 = \mathbf{U}^1(\Omega, S_v) = [\{\mathbf{u} \in \mathbf{C}^\infty(\overline{\Omega}) : \operatorname{div}\mathbf{u} = 0, \rho(\operatorname{supp}\mathbf{u}, \overline{S}_v) > 0\}]_{\mathbf{H}^1(\Omega)} \\ \mathbf{W}^1 = \mathbf{W}^1(\Omega, S_v) = \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{w} = 0, \mathbf{w}|_{S_v} = 0\} \end{gathered} \tag{3.1}$$

In the case $\partial\Omega = S_v$ the agreement between $\mathbf{U}^1$ and $\mathbf{W}^1$ is proved in /2/ (Theorem 2.2). Utilizing this, we obtain two assertions about the agreement between $\mathbf{U}^1$ and $\mathbf{W}^1$ that cover a sufficiently broad class (Ω, S_v).

The following auxiliary proposition is used in proving the first.

Lemma 3.1. Let Ω be a bounded strictly Lipschitzian domain; $\Gamma \subset \partial\Omega$; $\partial\Omega \setminus \Gamma$ contains a certain non-empty set open in $\partial\Omega$. Then for any function $\mathbf{u}$ from $\mathbf{H}^1(\Omega)$ with $\mathbf{u}|_\Gamma = 0$ there is a function $\mathbf{v}$ from $\mathbf{H}^1(\Omega)$ such that

$$\begin{gathered} \operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}|_\Gamma = 0 \\ \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leqslant c\|\operatorname{div}\mathbf{u}\|_{L_2(\Omega)} \end{gathered}$$

The validity of Lemma 3.1 follows directly from the results /2/.

Theorem 3.1. Let 1) Ω be a bounded strictly Lipschitzian domain; 2) $\overline{S}_q = \partial\Omega \cap \partial\Omega'$ where Ω' is a strictly Lipschitzian domain not intersecting Ω (Fig.1); 3) the domain G containing Ω and Ω' and such that $\overline{G} = \overline{\Omega} \cup \overline{\Omega}'$ is strictly Lipschitzian; 4) for every function $\mathbf{w}$ from $\mathbf{W}^1(\Omega, S_v)$ and for any $\varepsilon > 0$ there is a $\mathbf{w}_\varepsilon$ from $\mathbf{H}^1(\Omega)$ such that

$$\|\mathbf{w} - \mathbf{w}_\varepsilon\|_{\mathbf{H}^1(\Omega)} < \varepsilon, \quad \mathbf{w}_\varepsilon|_\Gamma = 0$$

where Γ is a certain neighborhood of $\overline{S}_v$ in $\partial\Omega$.

Then

$$\mathbf{U}^1(\Omega, S_v) = \mathbf{W}^1(\Omega, S_v)$$

Remarks 1°. This latter assumption is necessary; sufficient conditions for its satisfaction are given in Lemma 2.1.

2°. For $S_q = \varnothing$ the theorem is proved in /2/, it can henceforth be considered that $S_q \neq \varnothing$. We recall that conditions (0.1) are assumed satisfied all the time.

Proof. The embedding of $\mathbf{U}^1(\Omega, S_v)$ in $\mathbf{W}^1(\Omega, S_v)$ is evident. Let $\mathbf{w}$ now be a function from $\mathbf{W}^1(\Omega, S_v)$; we show that $\mathbf{w}$ belongs to $\mathbf{U}^1(\Omega, S_v)$.

We first consider the function $\mathbf{w}_\varepsilon$, whose existence is assured by the assumption 4. According to Lemma 3.1, a function $\mathbf{v}_\varepsilon$ from $\mathbf{H}^1(\Omega)$ exists such that

$$\begin{gathered} \operatorname{div}\mathbf{v}_\varepsilon = 0, \quad \mathbf{v}_\varepsilon|_\Gamma = 0 \\ \|\mathbf{w}_\varepsilon - \mathbf{v}_\varepsilon\|_{\mathbf{H}^1(\Omega)} < c\varepsilon \end{gathered} \tag{3.2}$$

(c is independent of $\mathbf{w}_\varepsilon$). It is later sufficient to confirm that $\mathbf{v}_\varepsilon$ can be approximated in $\mathbf{H}^1(\Omega)$ by a function from $\mathbf{U}^1(\Omega, S_v)$.

We construct a solenoidal continuation of the function $\mathbf{v}_\varepsilon$ in G that takes on a zero value on ∂G. To do this we consider first any continuation $\mathbf{V}_\varepsilon \in \mathbf{H}^1(R^n)$ of the function $\mathbf{v}_\varepsilon$ (it always exists for a strictly Lipschitzian domain Ω). We now note that since the compacts $\partial\Omega \setminus \Gamma$ and $\partial\Omega' \setminus S_q$ do not intersect, then there are nonintersecting neighborhoods $U(\partial\Omega \setminus \Gamma)$, $U(\partial\Omega' \setminus S_q)$ in R^n and a smooth function α finite in R^n, which takes on the value 1 in $U(\partial\Omega \setminus \Gamma)$ and the value 0 on $U(\partial\Omega' \setminus S_q)$. Then we have for the function $\alpha\mathbf{V}_\varepsilon$ from $\mathbf{H}^1(R^n)$

$$\alpha\mathbf{V}_\varepsilon|_{U(\partial\Omega'-S_q)} = 0, \quad \alpha\mathbf{V}_\varepsilon|_{\partial\Omega} = \mathbf{v}_\varepsilon|_{\partial\Omega} \tag{3.3}$$

The function $\mathbf{u}_\varepsilon$ which agrees with $\mathbf{v}_\varepsilon$ on Ω and with $\alpha\mathbf{V}_\varepsilon$ on $C\Omega$ belongs to $\mathbf{H}^1(R^n)$ and is a continuation of $\mathbf{v}_\varepsilon$.

We now consider the function $\mathbf{u}_\varepsilon' = \mathbf{u}_\varepsilon|_{\Omega'}$ from $\mathbf{H}^1(\Omega')$. By virtue of the relationships (3.3) and (3.2)

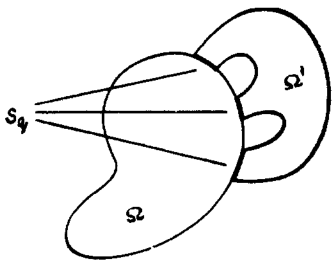

Fig.1

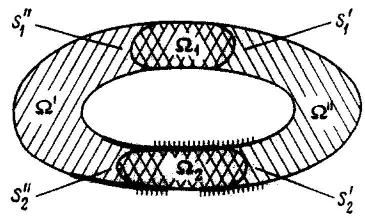

Fig.2

$$\int_{\partial\Omega'} \mathbf{u}_\varepsilon'\mathbf{v}'ds = \int_{S_q} \mathbf{u}_\varepsilon'\mathbf{v}'ds = -\int_{S_q} \mathbf{v}_\varepsilon\mathbf{v}ds = -\int_{\partial\Omega} \mathbf{v}_\varepsilon\mathbf{v}ds = 0$$

($\mathbf{v}, \mathbf{v}'$ are unit external normals to $\partial\Omega$, $\partial\Omega'$). Then there is /2/ a function $\mathbf{v}_\varepsilon'$ from $\mathbf{H}^1(\Omega')$ such that $\operatorname{div} \mathbf{v}_\varepsilon' = 0$, $\mathbf{v}_\varepsilon'|_{\partial\Omega'} = \mathbf{u}_\varepsilon'|_{\partial\Omega'}$.

Let $\mathbf{v}_\varepsilon^c$ be a function in G that agrees with $\mathbf{v}_\varepsilon$ on Ω and with $\mathbf{v}_\varepsilon'$ on Ω'. It is easy to see that $\mathbf{v}_\varepsilon^c \in \mathbf{H}^1(G)$ is a solenoidal continuation of $\mathbf{v}_\varepsilon$ on G that has a zero trace on ∂G. According to Theorem 2.2 from /2/, there then exists for $\mathbf{v}_\varepsilon^c$ an approximating solenoidal field from $\mathbf{C}_0^\infty(G)$. The limitation of this field on Ω evidently belongs to $\mathbf{U}^1(\Omega, \bar{S}_v)$ and approximates $\mathbf{v}_\varepsilon$ in $\mathbf{H}^1(\Omega)$, which proves the theorem.

Even in the smooth case, Theorem 3.1 is not applicable for every Ω, S_q. For instance, if the domain Ω on a plane has the shape of a ring, part of the set S_q is located on its inner circumference, and another part on its outer, then it is impossible to construct a domain Ω' satisfying the conditions of Theorem 3.1.

The agreement between $\mathbf{U}^1$ and $\mathbf{W}^1$ can be established in this and analogous cases by considering Ω and S_v as comprised of certain domains Ω', Ω'' and parts of their boundaries S_v', S_v''. Before proving a corresponding assertion, we list the requirements for the construction of a composite domain.

Let Ω, Ω', Ω'' be bounded domains in R^n, $\Omega = \Omega' \cup \Omega''$; Ω' is not embedded in Ω'', and Ω'' is not embedded in Ω'; the intersection $\Omega' \cap \Omega''$ consists of a finite number of domains $\Omega_i (i = 1, 2, \ldots, N)$, separated by positive distances; each of the domains Ω, Ω', Ω'', Ω_i is placed locally on one side of the boundary; a function α from $C^1(\bar{\Omega})$ exists on $\bar{\Omega}$ and takes on the value 0 on $\Omega \setminus \Omega'$ and the value 1 on $\Omega \setminus \Omega''$. Furthermore, let S_v, S_v', S_v'' be open subsets in $\partial\Omega$, $\partial\Omega'$ and $\partial\Omega''$, respectively; $S_v' \subset S_v$, $S_v'' \subset S_v$, $\bar{S}_v = \bar{S}_v' \cup \bar{S}_v''$; moreover, let $\bar{S}_v$ be contained in the union of $\bar{S}_v'$ with the boundary of the set $\Omega'' \setminus \Omega'$ and in the union of $\bar{S}_v''$ with the boundary of the set $\Omega' \setminus \Omega''$.

Upon compliance with these conditions, we call Ω and S_v regularly composed of Ω', Ω'' and S_v', S_v'' respectively. The listed requirements, although they appear awkward, describe a simple situation. An example is presented in Fig.2, where two domains Ω', Ω'' are shaded with different cross-hatchings, S_v' is depicted by a heavy line, and S_v'' is the dashed boundary. Moreover, parts of the boundaries $S_i' = \partial\Omega_i \cap \Omega''$, $S_i'' = \partial\Omega_i \cap \Omega'$ are indicated.

Let S' and S'' denote the unions US_i' and US_i'', respectively. It is easy to confirm that $S'(S'')$ is the intersection of $\partial\Omega'(\partial\Omega'')$ with the boundary of the set $\Omega'' \setminus \Omega' (\Omega' \setminus \Omega'')$.

Theorem 3.2. Let 1) Ω and S_v be regularly composed of Ω', Ω'' and S_v' and S_v'', respectively; 2) Ω, Ω_i $(i = 1, 2, \ldots, N)$ are strictly Lipschitzian domains, the boundary of the domain Ω_i contains a certain non-empty set U_i, $U_i \subset S_q$ in open $\partial\Omega_i$; 3) the following relationships are valid

$$\mathbf{U}^1(\Omega', S_v' \cup S') = \mathbf{W}^1(\Omega', S_v' \cup S') \tag{3.4}$$
$$\mathbf{U}^1(\Omega'', S_v'' \cup S'') = \mathbf{W}^1(\Omega'', S_v'' \cup S'')$$

Then $\mathbf{U}^1(\Omega, S_v) = \mathbf{W}^1(\Omega, S_v)$.

Proof. The embedding of $\mathbf{U}^1(\Omega, S_v)$ in $\mathbf{W}^1(\Omega, S_v)$ is evident. Now, let $\mathbf{w}$ belong to $\mathbf{W}^1(\Omega, S_v)$; we show that $\mathbf{w}$ belongs also to $\mathbf{U}^1(\Omega, S_v)$.

It is sufficient to see that $\mathbf{w}$ can be represented in the form $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$, where

$$\mathbf{w}' \in \mathbf{H}^1(\Omega), \quad \operatorname{div} \mathbf{w}' = 0, \quad \mathbf{w}'|_{S_v' \cup S'} = 0$$

and $\mathbf{w}'$ is the continuation to zero of a certain function from $\mathbf{H}^1(\Omega')$ in Ω, while $\mathbf{w}''$ possesses analogous properties with the replacement of the (′) by (″). Actually, because of the first of the conditions (3.4), for any $\varepsilon > 0$ a function $\mathbf{u}'$ from $\mathbf{C}^\infty(\overline{\Omega}')$ exists that vanishes is the neighborhood of $\bar{S}_v' \cup \bar{S}'$ in $\overline{\Omega}'$ such that

$$\operatorname{div} \mathbf{u}' = 0, \quad \|\mathbf{w}' - \mathbf{u}'\|_{\mathbf{H}^1(\Omega')} < \varepsilon$$

Since, as has been remarked above, $\bar{S}' = \partial\Omega' \cap \operatorname{Fr}(\Omega'' \setminus \Omega')$, then $\mathbf{u}'$ vanishes in the neighborhood of this set in Ω' ($\operatorname{Fr} M$ is the boundary of the set M in R^n). Then $\mathbf{u}_c'$, its continuation to zero in $\Omega'' \setminus \Omega'$ belongs to $\mathbf{C}^\infty(\overline{\Omega})$, where $\operatorname{div} \mathbf{u}_c' = 0$ and as is easily established, $\mathbf{u}_c'$ vanishes in the neighborhood of $\bar{S}_v$ in $\overline{\Omega}$. The function $\mathbf{u}_c''$ is constructed analogously. The following properties are then evident for $\mathbf{u} = \mathbf{u}_c' + \mathbf{u}_c''$

$$\mathbf{u} \in \mathbf{C}^\infty(\overline{\Omega}), \quad \operatorname{div} \mathbf{u} = 0$$
$$\rho(\operatorname{supp} \mathbf{u}, \bar{S}_v) > 0, \quad \|\mathbf{w} - \mathbf{u}\|_{\mathbf{H}^1(\Omega)} < 2\varepsilon$$

from which it is seen that $\mathbf{w}$ belongs to $\mathbf{U}^1(\Omega, S_v)$.

We now show that $\mathbf{w}$ can be represented in the requisite from $\mathbf{w} = \mathbf{w}' + \mathbf{w}''$.

We consider first the function $\mathbf{v} = \alpha\mathbf{w}$. Utilizing the properties of the function α we find that $\mathbf{v}$ belongs to $\mathbf{H}^1(\Omega)$ and is a continuation to zero of a certain function from $\mathbf{H}^1(\Omega')$ in Ω. Since $S' \subset \operatorname{Fr}(\Omega'' \setminus \Omega')$ and $\mathbf{w}|_{S_v} = 0$, then $\mathbf{v}|_{S_v \cup S'} = 0$. Compliance with the solenoidality condition should still be achieved, the function $\mathbf{v}$ should be rectified in the domains Ω_i $(i = 1, 2, \ldots, N)$ since only in them is $\operatorname{div} \mathbf{v} \neq 0$.

Let Γ_i be a neighborhood in $\partial\Omega_i$ of the set $\bar{S}_i' \cup \bar{S}_i'' \cup \bar{S}_{vi}$ $(S_{vi} = S_v \cap \partial\Omega_i)$, where $\partial\Omega_i \setminus \Gamma_i$ contains a certain set open in $\partial\Omega_i$ (Γ_i exists because of condition 2 of the theorem). Then there is a function

$$\mathbf{v}_i \in \mathbf{H}^1(\Omega_i), \quad \operatorname{div} \mathbf{v}_i = -\operatorname{div} \mathbf{v}, \quad \mathbf{v}_i|_{\Gamma_i} = 0$$

Now, let $\mathbf{w}_i$ be the continuation of $\mathbf{v}_i$ to zero in Ω. It is easy to verify that $\mathbf{w}_i|_{S_v \cup S' \cup S''} = 0$ and then the functions

$$\mathbf{w}' = \alpha\mathbf{w} + \sum_{i=1}^{N} \mathbf{w}_i, \quad \mathbf{w}'' = (1 - \alpha)\mathbf{w} - \sum_{i=1}^{N} \mathbf{w}_i$$

possess all the required properties. The theorem is proved.

We note that if $\mathbf{U}^1 = \mathbf{W}^1$, then evidently $\mathbf{U}^1 = \overline{\mathbf{V}}^1 = \mathbf{W}^1$ independently of whether the disappearance of $\mathbf{v}$ from $\mathbf{V}^1$ near or on $\bar{S}_v$ is required in the definition of $\mathbf{V}^1$.

4. Stokes formula. Later, as is mentioned in Sect.1, the Stokes formula is to be applied to the field τ for which not all the first derivatives are generally at least locally summable. In this connection, we consider the space of vector fields on Ω

$$\mathbf{K}(\Omega) = \{\mathbf{u} \in \mathbf{L}_2(\Omega) \colon \operatorname{div} \mathbf{u} \in L_2(\Omega)\}$$

with the norm

$$\|\mathbf{u}\|^2_{\mathbf{K}(\Omega)} = \|\mathbf{u}\|^2_{\mathbf{L}_2(\Omega)} + \|\operatorname{div} \mathbf{u}\|^2_{L_2(\Omega)}$$

The space $\mathbf{K}(\Omega)$ refers to the class of spaces $\mathbf{H}^M$ studied in /6/, where, however, the question about integration by parts in which we are interested was not examined. The space $\mathbf{K}(\Omega)$ is complete, where (for instance, in strictly Lipschitzian domains) $\mathbf{C}^\infty(\overline{\Omega})$ is compact.

Let Ω be a bounded domain of the class C^1 in R^n. Every function v from $C^\infty(\bar{\Omega})$ has a trace $v_\nu = \mathbf{v}|_{\partial\Omega}\nu$ on $\partial\Omega$ (ν is the unit external normal to $\partial\Omega$). It can be considered as an element of the space $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))'$, whose action on w from $H^{1/2}(\partial\Omega)$ is given by the relationship

$$\langle v_\nu, w\rangle = \int_{\partial\Omega} v_\nu w\, ds$$

We note that

$$\|v_\nu\|_{H^{-1/2}(\partial\Omega)} \leqslant c\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \tag{4.1}$$

where c is independent of $\mathbf{v}$. Indeed, we take its continuation w^c in Ω for w, as in (2.2). Then from the Stokes formula

$$\int_{\partial\Omega} v_\nu w\, ds = \int_\Omega w^c \operatorname{div} \mathbf{v}\, dx + \int_\Omega \mathbf{v} \operatorname{grad} \mathbf{w}^c\, dx$$

(4.1) follows. The completeness of $C^\infty(\bar{\Omega})$ in $\mathbf{K}(\Omega)$, the estimate (4.1), and the corresponding passage to the limit in the Stokes form now results in the following assertion.

Lemma 4.1. Let Ω be a bounded domain of the class C^1 in R^n. Then the mapping of the trace $\mathbf{u} \to \mathbf{u}_\nu$ from $\mathbf{K}(\Omega)$ in $H^{-1/2}(\partial\Omega)$

$$\langle u_\nu, w\rangle = \lim_{n\to\infty} \int_{\partial\Omega} u_\nu^{(n)} w\, ds \quad \forall w \in H^{1/2}(\partial\Omega)$$

(where $\{\mathbf{u}^{(n)}\}$ is any sequence of smooth functions converging to $\mathbf{u}$ in $\mathbf{K}(\Omega)$) is linear and continuous. For any w from $H^1(\Omega)$ and any $\mathbf{u}$ from $\mathbf{K}(\Omega)$ the Stokes formula is valid

$$\int_\Omega w \operatorname{div} \mathbf{u}\, dx = -\int_\Omega \mathbf{u} \operatorname{grad} w\, dx + \langle u_\nu, w|_{\partial\Omega}\rangle$$

5. Self-equilibrated stress fields in incompressible media. The plan noted in Sect.1 can now be realized. We first present an assertion about the relation between Σ^1 and Σ^2 in the case $\partial\Omega = S_v$, that results directly from results in /2/.

Theorem 5.1. (O.A. Ladyzhenskaia and V.A. Solonnikov). Let Ω be a bounded strictly Lipschitzian domain, $\partial\Omega = S_v$. Then for any τ from Σ^1 a pressure field $p \in L_2(\Omega)$ exists such that $\tau + p\mathbf{g}$ belongs to Σ^2.

Indeed, if $\tau_{ij} \in L_2(\Omega)$, then τ determines a linear continuous functional $\mathbf{f}_\tau$ in $H_0^1(\Omega)$

$$\langle \mathbf{f}_\tau, \mathbf{u}\rangle = -\int_\Omega \tau_{ij} \frac{\partial u_i}{\partial x^j}\, dx \quad (\forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)) \tag{5.1}$$

Then the Stokes problem is uniquely solvable /2/, i.e., there is a $\mathbf{v}$ from $\bar{\mathbf{V}}^1$ such that for any $\mathbf{u} \in \bar{\mathbf{V}}^1$

$$\int_\Omega \frac{\partial v_i}{\partial x^j} \frac{\partial u_i}{\partial x^j}\, dx = -\langle \mathbf{f}_\tau, \mathbf{u}\rangle$$

In the case under consideration $\tau \in \Sigma^1$ and, therefore, $\mathbf{f}_\tau|_{\bar{\mathbf{V}}^1} = 0$, hence $\mathbf{v} = 0$. Now we apply the Theorem 2.1 /2/ to this solution of the Stokes problem: there is a p from $L_2(\Omega)$ such that

$$\int_\Omega p \operatorname{div} \mathbf{w}\, dx = \langle \mathbf{f}_\tau, \mathbf{w}\rangle \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega)$$

Because of (5.1) this also means that $\tau + p\mathbf{g}$ belongs to Σ^2 (since $S_v = \partial\Omega$ in the case under consideration and, therefore, $\bar{\mathbf{V}}^2 = \mathbf{H}_0^1(\Omega)$).

Furthermore, we consider the case of mixed boundary conditions. Let $\partial\Omega \neq S_v$ and τ belong to the set $\Sigma^1 = \Sigma^1(\Omega, S_v)$.

First step. Since $\Sigma^1(\Omega, S_v)$ is evidently embedded in $\Sigma^1(\Omega, \partial\Omega)$, then according to Theorem 5.1 a $p^0 \in L_2(\Omega)$ exists such that for $\tau^\circ = \tau + p^\circ\mathbf{g}$ the following relationships are satisfied

$$\partial\tau_{ij}^\circ/\partial x^j = 0. \tag{5.2}$$

Second step. By virtue of this latter relationship, the vectors $\tau_{(i)}^\circ$ $(i = 1, 2, \ldots, n)$ with the components $\{\tau_{i1}^\circ, \tau_{i2}^\circ, \ldots, \tau_{in}^\circ\}$ belong to the space $\mathbf{K}(\Omega)$. Then in the relationships

$$\int_\Omega \tau_{ij}^\circ \frac{\partial v_i}{\partial x^j}\, dx = 0$$

(satisfied for every $\mathbf{v}$ from $\bar{\mathbf{V}}^1(\Omega, S_v)$ since τ° together with τ belongs to $\Sigma^1(\Omega, S_v)$) the Stokes formula can be used (Lemma 4.1). Taking account of (5.2), we find that for any $\mathbf{v}$ from $\bar{\mathbf{V}}^1$

$$\langle \gamma, \mathbf{v}|_{\partial\Omega}\rangle = 0 \tag{5.3}$$

where γ belongs to $\mathbf{H}^{-1/2}(\partial\Omega)$ and has the components $\gamma_i = \tau^\circ_{(i)\nu}$.

Third step. Now, let $\mathbf{u}$ be any field from $\bar{\mathbf{V}}^2$, and $\mathbf{u}_0$ some field from $\bar{\mathbf{V}}^2$ for which

$$\mathbf{u}_0|_{S_v} = 0, \quad \int_{\partial\Omega} \mathbf{u}_0 \nu\, ds = 1$$

(there is such a field since $\partial\Omega \neq S_v$). Then for

$$\mathbf{v} = \mathbf{u} - \mathbf{u}_0 \int_{\partial\Omega} \mathbf{u}\nu\, ds \tag{5.4}$$

the following relations are satisfied

$$\mathbf{v} \in \mathbf{H}^1(\Omega), \quad \mathbf{v}|_{S_v} = 0, \quad \int_{\partial\Omega} \mathbf{v}\nu\, ds = 0 \tag{5.5}$$

Furthermore, we consider the field $\mathbf{v}_s$ that possesses the following properties

$$\mathbf{v}_s \in \mathbf{H}^1(\Omega), \quad \operatorname{div} \mathbf{v}_s = 0, \quad \mathbf{v}_s|_{\partial\Omega} = \mathbf{v}|_{\partial\Omega}$$

(because of the conditions (5.5) such a field is in /2/). We note that $\mathbf{v}_s$ belongs to $\mathbf{W}^1(\Omega, S_v)$.

If $\mathbf{W}^1(\Omega, S_v)$ agrees with $\bar{\mathbf{V}}^1(\Omega, S_v)$, then by virtue of (5.3) $\langle \gamma, \mathbf{v}_s|_{\partial\Omega}\rangle = 0$ or, equivalently, $\langle \gamma, \mathbf{v}|_{\partial\Omega}\rangle = 0$.

According to (5.4), this means that for any $\mathbf{u}$ from $\bar{\mathbf{V}}^2(\Omega, S_v)$

$$\langle \gamma, \mathbf{u}\rangle = c_0 \int_{\partial\Omega} \mathbf{u}\nu\, ds, \quad c_0 = \langle \gamma, \mathbf{u}_0|_{\partial\Omega}\rangle \tag{5.6}$$

We set $\sigma = \tau^\circ - c_0 \mathbf{g}$. Using the Stokes formula and (5.2), we find that

$$\int_\Omega \sigma_{ij} \frac{\partial u_i}{\partial x^j}\, dx = \langle \gamma, \mathbf{u}|_{\partial\Omega}\rangle - \langle c_0, \nu, \mathbf{u}|_{\partial\Omega}\rangle \ \forall \mathbf{u} \in \bar{\mathbf{V}}^2(\Omega, S_v)$$

As follows from (5.6), the right side vanishes here, and therefore, $\sigma = \tau + (p^\circ - c_0)\mathbf{g}$ belongs to $\Sigma^2(\Omega, S_v)$. The following assertion is thereby proved.

Theorem 5.2. Let Ω be a bounded domain of class C^1; $S_q \neq \varnothing$; $\mathbf{W}^1(\Omega, S_v) = \mathbf{V}^1(\Omega, S_v)$. Then for any τ from $\Sigma^1(\Omega, S_v)$ there is a pressure field $p \in L_2(\Omega)$ such that $\tau + p\mathbf{g}$ belongs to $\Sigma^2(\Omega, S_v)$.

Remarks. 1°. Sufficient conditions for the agreement between $\mathbf{W}^1$ and $\bar{\mathbf{V}}^1$ are given by Theorems 3.1 and 3.2.

2°. The pressure field is defined uniquely by the deviator component of τ. More exactly, if τ_1 and τ_2 belong to Σ^1 and their deviator components agree, but $\sigma_1 = \tau_1 + p_1\mathbf{g}$ and $\sigma_2 = \tau_2 + p_2\mathbf{g}$ belong to Σ^2, then $\sigma_1 = \sigma_2$ for $S_q \neq \varnothing$ and $\sigma_1 - \sigma_2 = c\mathbf{g}$, where c is an arbitrary constant, for $S_q = \varnothing$.

Certain problems of the mechanics of incompressible media reduce to the problem of finding a self-equilibrated field of stresses σ that satisfies definite conditions. If these conditions do not impose constraints on the spherical component of σ, then it can generally be eliminated from consideration by comprehending self-equilibration as belonging of the field of stresses σ to the set Σ^1. In such a "deviator" problem the spherical component of the required stresses is not determined (we recall that every τ from Σ^1 is defined to the accuracy of the addition of an arbitrary spherical tensor field).

The solution of the complete problem (in whose formulation the self-equilibration of σ is understood as belonging of the σ to the set Σ^2) is also a solution of the deviator problem. Theorem 5.2 can be used for the reverse comparison of the solution $\tau + pg$ of the total problem to the solution τ, of the deviator problem.

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Translated by M.D.F.